3-5 Systems in Three Variables

TEKS FOCUS

TEKS (3)(B) Solve systems of three linear equations in three variables by using Gaussian elimination, technology with matrices, and substitution.

TEKS (1)(D) Communicate mathematical ideas, reasoning, and their implications using multiple **representations**, including symbols, diagrams, graphs, and language as appropriate.

Additional TEKS (1)(A), (3)(A)

VOCABULARY

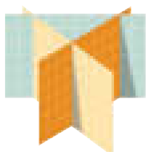
- **Representation** – a way to display or describe information. You can use a representation to present mathematical ideas and data.

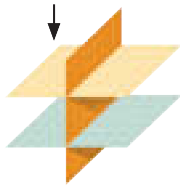
ESSENTIAL UNDERSTANDING

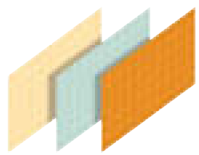
To solve systems of three equations in three variables, you can use some of the same algebraic methods you used to solve systems of two equations in two variables.

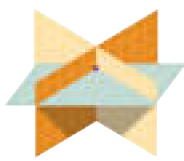
take note — Key Concept Solutions of Systems With Three Variables

No solution
No point lies in all three planes.



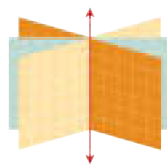






One solution
The planes intersect at one point.

An infinite number of solutions
The planes intersect at all the points along a common line.



Problem 1

Solving a System Using Elimination

What is the solution of the system? Use elimination. The equations are numbered to make the procedure easy to follow.

$$\begin{cases} 2x - y + z = 4 & \text{①} \\ x + 3y - z = 11 & \text{②} \\ 4x + y - z = 14 & \text{③} \end{cases}$$

Think

Which variable do you eliminate first? Eliminate the variable for which the process requires the fewest steps.

Step 1 Pair the equations to eliminate z. Then you will have two equations in x and y.

Add.

$$\begin{array}{ll} \text{①} & 2x - y + z = 4 \\ \text{②} & x + 3y - z = 11 \\ \hline \text{④} & 3x + 2y \quad\;\; = 15 \end{array}$$

Subtract.

$$\begin{array}{ll} \text{②} & x + 3y - z = 11 \\ \text{③} & 4x + y - z = 14 \\ \hline \text{⑤} & -3x + 2y \quad\;\; = -3 \end{array}$$

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Problem 1 *continued*

Step 2 Write the two new equations as a system. Solve for x and y.

Add and solve for y.

$$\begin{aligned} ④\ & 3x + 2y = 15 \\ ⑤\ & -3x + 2y = -3 \\ \hline & 4y = 12 \\ & y = 3 \end{aligned}$$

Substitute $y = 3$ and solve for x.

$$\begin{aligned} ④\quad 3x + 2y &= 15 \\ 3x + 2(3) &= 15 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Think

Does it matter which equation you substitute into to find *z*?

No, you can substitute into any of the original three equations.

Step 3 Solve for z. Substitute the values of x and y into one of the original equations.

① $2x - y + z = 4$	Use equation ①.
$2(3) - 3 + z = 4$	Substitute.
$6 - 3 + z = 4$	Simplify.
$z = 1$	Solve for z.

Step 4 Write the solution as an ordered triple. The solution is (3, 3, 1).

Problem 2

TEKS Process Standard (1)(D)

Solving an Equivalent System

What is the solution of the system? Use elimination.

$$\begin{cases} x + y + 2z = 3 & ① \\ 2x + y + 3z = 7 & ② \\ -x - 2y + z = 10 & ③ \end{cases}$$

Think	Write
You are trying to get two equations in x and z. Multiply ① so you can add it to ② and eliminate y. Do the same with ② and ③.	① $x + y + 2z = 3$, ② $2x + y + 3z = 7$ $\longrightarrow$ $-x - y - 2z = -3$; $2x + y + 3z = 7$; ④ $x + z = 4$ ② $2x + y + 3z = 7$, ③ $-x - 2y + z = 10$ $\longrightarrow$ $4x + 2y + 6z = 14$; $-x - 2y + z = 10$; ⑤ $3x + 7z = 24$
Multiply ④ so you can add it to ⑤ and eliminate x.	④ $x + z = 4$, ⑤ $3x + 7z = 24$ $\longrightarrow$ $-3x - 3z = -12$; $3x + 7z = 24$; $4z = 12$; $z = 3$
Substitute $z = 3$ into ④. Solve for x.	$x + 3 = 4$ $x = 1$
Substitute the values for x and z into ① to find y. Check the answer in the three original equations.	$x + y + 2z = 3$ $1 + y + 2(3) = 3$ $y = -4$ Check $1 + (-4) + 2(3) = 3$ ✔ $2(1) + (-4) + 3(3) = 7$ ✔ $-(1) - 2(-4) + 3 = 10$ ✔ The solution is $(1, -4, 3)$.

Problem 3

Solving a System Using Substitution

Multiple Choice What is the x-value in the solution of the system?

$$\begin{array}{l} ① \\ ② \\ ③ \end{array}\begin{cases} 2x + 3y - 2z = -1 \\ x + 5y = 9 \\ 4z - 5x = 4 \end{cases}$$

Ⓐ 1 Ⓑ 4 Ⓒ 6 Ⓓ 10

Think

Which equation should you solve for one of its variables? Look for an equation that has a variable with coefficient 1.

Step 1 Choose equation ②. Solve for x.

② $x + 5y = 9$

$x = 9 - 5y$

Step 2 Substitute the expression for x into equations ① and ③ and simplify.

①
$$\begin{aligned} 2x + 3y - 2z &= -1 \\ 2(9 - 5y) + 3y - 2z &= -1 \\ 18 - 10y + 3y - 2z &= -1 \\ 18 - 7y - 2z &= -1 \\ -7y - 2z &= -19 \end{aligned}$$
④ $-7y - 2z = -19$

③
$$\begin{aligned} 4z - 5x &= 4 \\ 4z - 5(9 - 5y) &= 4 \\ 4z - 45 + 25y &= 4 \\ 4z + 25y &= 49 \\ 25y + 4z &= 49 \end{aligned}$$
⑤ $25y + 4z = 49$

Step 3 Write the two new equations as a system. Solve for y and z.

$$\begin{array}{l} ④ \\ ⑤ \end{array}\begin{cases} -7y - 2z = -19 \\ 25y + 4z = 49 \end{cases}$$

$$\begin{aligned} -14y - 4z &= -38 \\ 25y + 4z &= 49 \\ \hline 11y &= 11 \\ y &= 1 \end{aligned}$$

Multiply by 2. Then add.

④
$$\begin{aligned} -7y - 2z &= -19 \\ -7(1) - 2z &= -19 \\ -2z &= -12 \\ z &= 6 \end{aligned}$$

Substitute the value of y into ④.

Step 4 Use one of the original equations to solve for x.

②
$$\begin{aligned} x + 5y &= 9 \\ x + 5(1) &= 9 \\ x &= 4 \end{aligned}$$

Substitute the value of y into ②.

The solution of the system is (4, 1, 6), and $x = 4$.

The correct answer is B.

Problem 4

TEKS Process Standard (1)(A)

Solving a Real-World Problem

Business **You manage a clothing store and budget \$6000 to restock 200 shirts. You can buy T-shirts for \$12 each, polo shirts for \$24 each, and rugby shirts for \$36 each. If you want to have twice as many rugby shirts as polo shirts, how many of each type of shirt should you buy?**

Think

How many unknowns are there?
There are three unknowns: the number of each type of shirt.

Relate

T-shirts + polo shirts + rugby shirts = 200

rugby shirts = 2 • polo shirts

12 • T-shirts + 24 • polo shirts + 36 • rugby shirts = 6000

Define

Let x = the number of T-shirts.

Let y = the number of polo shirts.

Let z = the number of rugby shirts.

Write

$$\begin{cases} x + y + z = 200 & (1) \\ z = 2 \cdot y & (2) \\ 12 \cdot x + 24 \cdot y + 36 \cdot z = 6000 & (3) \end{cases}$$

Step 1 Since 12 is a common factor of all the terms in equation ③, write a simpler equivalent equation.

$$\begin{cases} 12x + 24y + 36z = 6000 & (3) \\ x + 2y + 3z = 500 & (4) \end{cases}$$

Divide by 12.

Step 2 Substitute $2y$ for z in equations ① and ④. Simplify to find equations ⑤ and ⑥.

① $x + y + z = 200$
$x + y + (2y) = 200$
⑤ $x + 3y = 200$

④ $x + 2y + 3z = 500$
$x + 2y + 3(2y) = 500$
⑥ $x + 8y = 500$

Step 3 Write ⑤ and ⑥ as a system. Solve for x and y.

$$\begin{cases} x + 3y = 200 & (5) \\ x + 8y = 500 & (6) \end{cases}$$

$$\begin{aligned} -x - 3y &= -200 \\ x + 8y &= 500 \\ \hline 5y &= 300 \\ y &= 60 \end{aligned}$$

Multiply by −1.
Then add.

⑤ $x + 3y = 200$
$x + 3(60) = 200$
$x = 20$

Substitute the value of y into ⑤.

Step 4 Substitute the value of y in ② and solve for z.

② $z = 2y$
$z = 2(60) = 120$

You should buy 20 T-shirts, 60 polo shirts, and 120 rugby shirts.

PRACTICE and APPLICATION EXERCISES

Scan page for a Virtual Nerd™ tutorial video.

For additional support when completing your homework, go to **PearsonTEXAS.com**.

Solve each system by elimination. Check your answers.

1. $\begin{cases} x - y + z = -1 \\ x + y + 3z = -3 \\ 2x - y + 2z = 0 \end{cases}$

2. $\begin{cases} x - y - 2z = 4 \\ -x + 2y + z = 1 \\ -x + y - 3z = 11 \end{cases}$

3. $\begin{cases} -2x + y - z = 2 \\ -x - 3y + z = -10 \\ 3x + 6z = -24 \end{cases}$

4. $\begin{cases} a + b + c = -3 \\ 3b - c = 4 \\ 2a - b - 2c = -5 \end{cases}$

5. $\begin{cases} 6q - r + 2s = 8 \\ 2q + 3r - s = -9 \\ 4q + 2r + 5s = 1 \end{cases}$

6. $\begin{cases} x - y + 2z = -7 \\ y + z = 1 \\ x = 2y + 3z \end{cases}$

7. $\begin{cases} x + 2y = 2 \\ 2x + 3y - z = -9 \\ 4x + 2y + 5z = 1 \end{cases}$

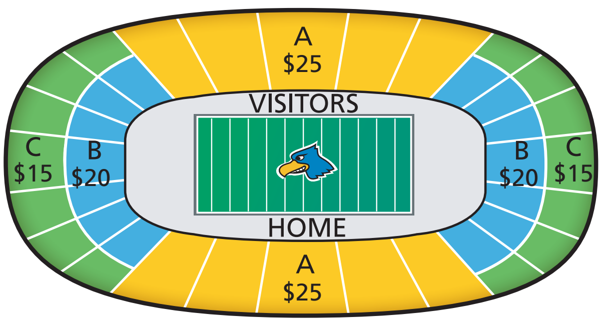
8. $\begin{cases} 3x + 2y + 2z = -2 \\ 2x + y - z = -2 \\ x - 3y + z = 0 \end{cases}$

9. $\begin{cases} x + 4y - 5z = -7 \\ 3x + 2y + 3z = 7 \\ 2x + y + 5z = 8 \end{cases}$

STEM **10. Apply Mathematics (1)(A)** In a factory there are three machines, *A, B,* and *C*. When all three machines are working, they produce 287 bolts per hour. When only machines *A* and *C* are working, they produce 197 bolts per hour. When only machines *A* and *B* are working, they produce 202 bolts per hour. How many bolts can each machine produce per hour?

11. In $\triangle PQR$, the measure of angle Q is three times that of angle P. The measure of angle R is 20° more than that of angle P. Find the measure of each angle.

12. Apply Mathematics (1)(A) A stadium has 49,000 seats. The number of seats in Section A equals the total number of seats in Sections B and C. Suppose the stadium takes in $1,052,000 from each sold-out event. How many seats does each section hold?



Solve each system by substitution. Check your answers.

13. $\begin{cases} x + 2y + 3z = 6 \\ y + 2z = 0 \\ z = 2 \end{cases}$

14. $\begin{cases} 3a + b + c = 7 \\ a + 3b - c = 13 \\ b = 2a - 1 \end{cases}$

15. $\begin{cases} 5r - 4s - 3t = 3 \\ t = s + r \\ r = 3s + 1 \end{cases}$

16. $\begin{cases} 13 = 3x - y \\ 4y - 3x + 2z = -3 \\ z = 2x - 4y \end{cases}$

17. $\begin{cases} x + 3y - z = -4 \\ 2x - y + 2z = 13 \\ 3x - 2y - z = -9 \end{cases}$

18. $\begin{cases} x - 4y + z = 6 \\ 2x + 5y - z = 7 \\ 2x - y - z = 1 \end{cases}$

Solve each system using any method.

19. $\begin{cases} x - 3y + 2z = 11 \\ -x + 4y + 3z = 5 \\ 2x - 2y - 4z = 2 \end{cases}$

20. $\begin{cases} x + 2y + z = 4 \\ 2x - y + 4z = -8 \\ -3x + y - 2z = -1 \end{cases}$

21. $\begin{cases} 4x - y + 2z = -6 \\ -2x + 3y - z = 8 \\ 2y + 3z = -5 \end{cases}$

22. $\begin{cases} 4x - 2y + 5z = 6 \\ 3x + 3y + 8z = 4 \\ x - 5y - 3z = 5 \end{cases}$

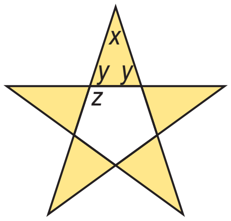
23. $\begin{cases} 2\ell + 2w + h = 72 \\ \ell = 3w \\ h = 2w \end{cases}$

24. $\begin{cases} 6x + y - 4z = -8 \\ \frac{y}{4} - \frac{z}{6} = 0 \\ 2x - z = -2 \end{cases}$

25. Apply Mathematics (1)(A) A worker received a \$10,000 bonus and decided to split it among three different accounts. He placed part in a savings account paying 4.5% per year, twice as much in government bonds paying 5%, and the rest in a mutual fund that returned 4%. His income from these investments after one year was \$455. How much did the worker place in each account?

26. Connect Mathematical Ideas (1)(F) Write your own system with three variables. Begin by choosing the solution. Then write three equations that are true for your solution. Use elimination to solve the system.

27. Refer to the regular five-pointed star at the right. Write and solve a system of three equations to find the measure of each labeled angle.



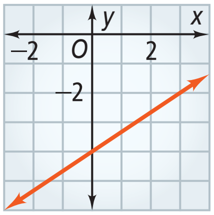
28. In the regular polyhedron described below, all faces are congruent polygons. Use a system of three linear equations to find the numbers of vertices, edges, and faces.

Every face has five edges and every edge is shared by two faces. Every face has five vertices and every vertex is shared by three faces. The sum of the number of vertices and faces is two more than the number of edges.

TEXAS Test Practice

29. What is the value of z in the solution of the system? $\begin{cases} y = -2x + 10 \\ -x + y - 2z = -2 \\ 3x - 2y + 4z = 7 \end{cases}$

30. What is the x-intercept of the line at the right after it is translated up 3 units?



31. Suppose y varies directly with x, and $y = 15$ when $x = 10$. What is y when $x = 22$?

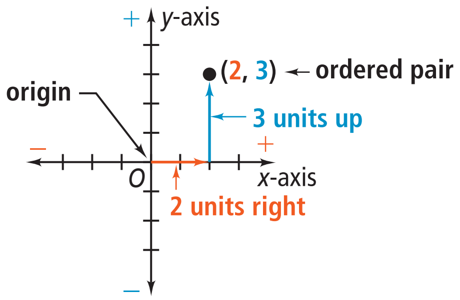
32. A theater has 490 seats. Seats sell for \$25 on the floor, \$20 in the mezzanine, and \$15 in the balcony. The number of seats on the floor equals the total number of seats in the mezzanine and balcony. Suppose the theater takes in \$10,520 from each sold-out event. How many seats does the mezzanine section hold?

Activity Lab | Graphs in Three Dimensions

USE WITH LESSON 3-5 TEKS (1)(A)

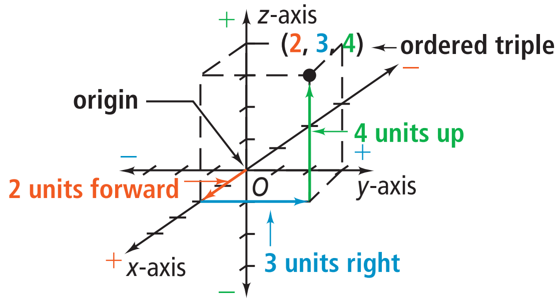
To describe positions in space, you need a three-dimensional coordinate system. You have learned to graph on an *xy*-coordinate plane using ordered pairs. Adding a third axis, the *z*-axis, to the *xy*-coordinate plane creates **coordinate space**. In coordinate space you graph points using **ordered triples** of the form (x, y, z).

Points in a Plane



A two-dimensional coordinate system allows you to graph points in a plane.

Points in Space



A three-dimensional coordinate system allows you to graph points in space.

In the coordinate plane, point (2, 3) is two units right and three units up from the origin. In coordinate space, point (2, 3, 4) is two units forward, three units right, and four units up.

Activity 1

Define one corner of your classroom as the origin of a three-dimensional coordinate system like the classroom shown. Write the coordinates of each item in your coordinate system.

1. each corner of your classroom
2. each corner of your desk
3. one corner of the blackboard
4. the clock
5. the waste-paper basket
6. Pick 3 items in your classroom and write the coordinates of each.

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An equation in two variables represents a line in a plane. An equation in three variables represents a plane in space.

Activity 2

Given the following equation in three variables, draw the plane in a coordinate space. $x + 2y - z = 6$

7. Let $x = 0$. Graph the resulting equation in the yz-plane.

8. Let $y = 0$. Graph the resulting equation in the xz-plane.

From geometry you know that two non-skew lines determine a plane.

9. Sketch the plane $x + 2y - z = 6$.
(If you need help, find a third line by letting $z = 0$ and then graph the resulting equation in the xy-plane.)

Activity 3

Two equations in three variables represent two planes in space.

10. Draw the two planes determined by the following equations:
$2x + 3y - z = 12$
$2x - 4y + z = 8$

11. Describe the intersection of the two planes above.

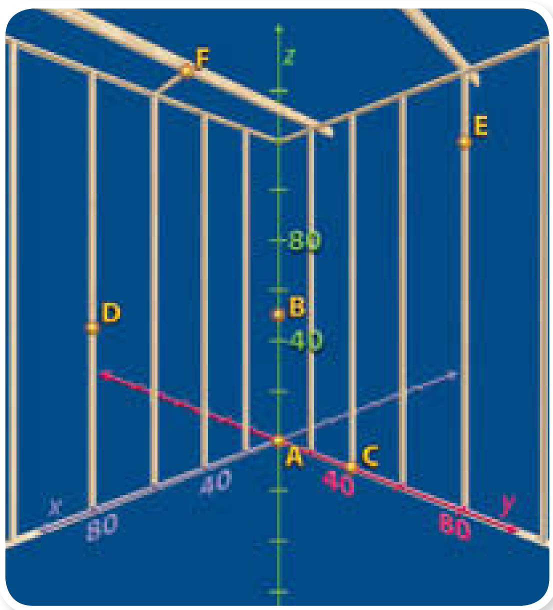
Exercises

Find the coordinates of each point in the diagram.

12. A **13.** B

14. C **15.** D

16. E **17.** F



Sketch the graph of each equation.

18. $x - y - 4z = 8$ **19.** $x + y + z = 2$

20. $-3x + 5y + 10z = 15$ **21.** $6x + 6y - 12z = 36$

Graph the following pairs of equations in the same coordinate space and describe their intersection, if any.

22. $-x + 3y + z = 6$
$-3x + 5y - 2z = 60$

23. $-2x - 3y + 5z = 7$
$2x - 3y - 4z = -4$